

## Subharmonic motion of particles in a vibrating tube

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We study the motion of strongly inelastic particles in a narrow vibrating tube using molecular-dynamics simulation. At low frequency of the vibration, we observe qualitative changes of the motion as the depth of the pile increases. The center of mass of the particle cloud can be described by a superposition of modes of different frequencies. For certain values of the depth, a single mode dominates. The frequency of the dominant mode is  $1/2$ ,  $1/3$ , or  $1/1$  of the vibration. We suggest that the behavior can be understood in terms of a new time scale  $\tau$ , reflecting the recompaction time for a finite-depth pile. [S1063-651X(98)50208-5]

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Granular material under vibration has been a constant source of interesting phenomena, such as convection, heap formation, and surface waves [1–5]. When the depth is sufficiently small (a shallow bed), the motion of a pile under vibration is qualitatively the same as that of a single layer. As the depth increases, changes are expected, which raises several interesting questions. For example, at what depth does the motion start to significantly differ from that of a single layer? Do the changes occur gradually or suddenly with an increase of the depth?

These questions are also closely related to some of proposed mechanisms for parametric waves and convection cells observed in vibrated granular material. For example, an estimate of the onset of the parametric waves, given in Ref. [6], is in excellent agreement with experiment. The estimation is based on the motion of a single particle. Also related is an argument for convection, which involves a bifurcation from a single-particle motion [7]. A systematic study of the depth dependence of the motion is necessary to check the validity and limitations of such mechanisms.

There have been several studies on the state of granular material under vibration. Clément and Rajchenbach measured the density, velocity, and temperature fields of the two-dimensional packing of beads [8]. Luding *et al.* studied the density field of the one- and two-dimensional systems, and found a different scaling [9], which was confirmed by Warr *et al.* [10] and Lee [11]. Lan and Rosato measured the density and temperature fields of the three-dimensional system [12].

While these works focus on the dependence of the fields on the frequency and amplitude of the vibration, there are few works specifically on the effect of the depth of a pile. Thomas *et al.* studied the system in three dimensions, focusing on a shallow bed [13], and found four distinct behaviors as the depth increases. They are (1) “Newtonian-I,” where the particles are bouncing so randomly that there is little change in the density field during a cycle; (2) “Newtonian-II,” where a dense layer of particles forms during one part of each cycle; (3) “coherent-expanded,” where the particles move as a coherent mass; and (4) “coherent-condensed,”

where the particles move as a coherent mass but remain compact through a cycle. Also, Brennen *et al.* studied the effect of the depth as well as the amplitude on the expansion of a pile [14]. They found at least one sudden change in the expansion at a certain amplitude, whose value changes with the depth.

In this paper, we systematically study the depth dependence for various combinations of the amplitude and frequency of the vibration. We use the molecular-dynamics (MD) simulation method, which provides detailed information on the motion of individual particles as well as time-averaged fields.

In the experiment of Thomas *et al.*, the motion of a pile approaches that of a single block as the depth increases [13]. We are interested to know whether this is the only possible dependence. Here, we choose to use strongly inelastic particles. By doing so, we can study the motion in the other extreme, since the typical value of  $e$  used in experiments is rather large [15]. Here,  $e$  is the coefficient of restitution between the particles. Furthermore, one may argue that a pile with large  $e$  behaves like a shorter pile with small  $e$ , since a pile with large  $e$  can be divided into blocks, each of which acts like a single particle.

We find that the system shows rich and unexpected dependence on the depth. In general, several modes of different frequencies are necessary to describe the resulting motion. However, at specific values of the depth, one of them dominates. The dominant mode is, besides single-particle motion, always a subharmonic ( $1/2$  or  $1/3$ ) of the frequency of vibration. We suggest that these behaviors result from an additional recompaction time scale  $\tau$  introduced in the system. When an initially compact pile is launched from the bottom, it takes time  $\tau$  to be compact again after subsequent collisions. If  $\tau$  becomes comparable to or even larger than the period  $T$  of the vibration, we expect the motion of the pile to be significantly different from that of a single layer. A dominant mode seems to occur when  $\tau$  is close to an integer multiple of  $T$ . At a somewhat higher frequency (100 Hz), a pile moves as a single block and  $\tau$  becomes negligible.

The simulations are done in two dimensions with disk-shaped particles, using a form of interaction due to Cundall and Strack [16,17]. Particles interact only by contact, and the force between two such particles  $i$  and  $j$  is the following. Let

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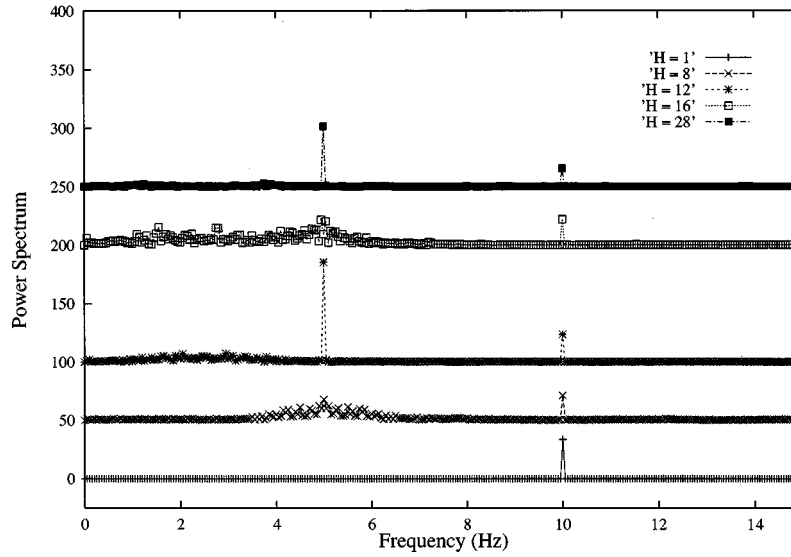


FIG. 1. Power spectra  $P(f)$  of the center of mass of the particles. The five curves correspond, from bottom to top, to  $H=1, 8, 12, 16,$  and  $28$ . The curves have been offset for clarity. Here,  $f=10$  and  $\Gamma=2$ .

the coordinate of the center of particle  $i$  ( $j$ ) be  $\vec{R}_i$  ( $\vec{R}_j$ ), and  $\vec{r} = \vec{R}_i - \vec{R}_j$ . The normal component  $F_{j \rightarrow i}^n$  of the force acting on particle  $i$  from particle  $j$  is

$$F_{j \rightarrow i}^n = k_n(a_i + a_j - |\vec{r}|) - \gamma m_e(\vec{v} \cdot \hat{n}), \quad (1)$$

where  $a_i$  ( $a_j$ ) is the radius of particle  $i$  ( $j$ ), and  $\vec{v} = d\vec{r}/dt$ . Here,  $k_n$  is the elastic constant,  $\gamma$  the friction coefficient, and  $m_e$  is the effective mass,  $m_i m_j / (m_i + m_j)$ . The shear component  $F_{j \rightarrow i}^s$  is given by

$$F_{j \rightarrow i}^s = -\text{sgn}(\delta s) \min(k_s |\delta s|, \mu |F_{j \rightarrow i}^n|), \quad (2)$$

where  $\mu$  is the friction coefficient,  $\delta s$  the total shear displacement during a contact, and  $k_s$  is the elastic constant of a virtual tangential spring. The shear force applies a torque to the particles, which then rotate.

Particles can also interact with walls. The force and torque on particle  $i$  in contact with a wall are given by (1)–(2) with  $a_j=0$  and  $m_e=m_i$ . Also, the system is in a vertical gravitational field  $\vec{g}$ . The interaction parameters used in this study are fixed as follows, unless otherwise specified:  $k_n = k_s = 5 \times 10^4$ ,  $\gamma = 10^3$ , and  $\mu = 0.2$ . In order to avoid artifacts of a monodisperse system (e.g., hexagonal packing), we choose the radius of the particles from a Gaussian distribution with the mean 0.1 and the width 0.02. The density of the particles is 5. Throughout this paper, cgs units are implied.

We put the particles into a two-dimensional rectangular box. The box consists of two horizontal (top and bottom) plates which oscillate sinusoidally along the vertical direction with given amplitude  $A$  and frequency  $f$ . The width and height of the box is 1 and  $10^4$ , respectively. The small width is used to suppress the surface waves [6]. We apply a periodic boundary condition in the horizontal direction.

The coefficient of restitution between the particles  $e_{pp}$ , determined from the above interaction parameters, is  $8.0 \times 10^{-2}$ , and the coefficient between the particles and the wall is  $e_{pw} = 2.5 \times 10^{-3}$ . The particles are thus almost completely inelastic. We have studied the motion of a single

particle for several values of  $A$  with  $f=10$ , and find good agreement in the predictions of Mehta and Luck [18].

We measure the time series  $Y(t)$  of the center of mass of the particles and compare it with that of a single particle. In Fig. 1, we show the power spectrum  $P(f)$  of the series for several values of  $H$ . Here,  $f=10$ ,  $\Gamma \equiv A(2\pi f)^2/g = 2$ , and  $H$ , the total number of layers in a pile, is 1, 8, 12, 16, and 28 (from bottom to top). The measurements are made for 200 cycles. The motion of a single particle with these parameters is known to have the same period as the vibration [19]. This is confirmed by the fact that  $P(f)$  with  $H=1$  is strongly dominated by the mode at  $f=10$ .

For larger values of  $H$ , however, the behavior becomes quite different. As  $H$  increases, the  $f=10$  mode becomes less dominant ( $H \sim 8$ ), and then a  $1/2$  subharmonic mode becomes dominant ( $H \sim 12$ ). By further increasing  $H$ , no clear dominant mode is present ( $H=16$ ), and a  $1/2$  subharmonic mode dominates again ( $22 < H < 30$ ). Thus several modes are always present in the spectrum, and one of them dominates around specific values of  $H$ . We find that these qualitative features of the power spectrum seem to be insensitive to small changes of the width, coefficients of restitution, and elastic constant.

We now investigate the mechanism for the behaviors. We start with the observation that the particles in a box do not remain as a single block, but tend to be dispersed. We introduce quantity  $R(t)$ , which characterizes the dispersion as  $R(t) \equiv \sqrt{\langle y^2 \rangle - \langle y \rangle^2}$ . Here  $y_i(t)$  is the vertical coordinate of particle  $i$  and the average is taken over the particles. We describe the motion of the pile using their effective center  $Y(t)$  and effective “radius”  $R(t)$ .

The radius  $R(t)$  does vary with time. It remains small, while the particles are resting on the bottom. When they are launched into the air,  $R(t)$  initially increases, then decreases after they collide with the bottom. This procedure introduces additional time scale  $\tau$ , the time needed for a compact pile to be compact again after launching and subsequent landing(s). One can think of  $\tau$  as “relaxation time” for the pile. When  $\tau$

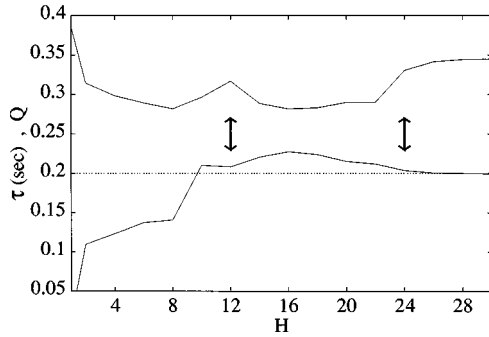


FIG. 2. Quantity  $Q$  (top) that characterizes the dominance of a single mode and the relaxation time  $\bar{\tau}$  (bottom) are shown for several values of  $H$ . The curve of  $Q$  has been offset and rescaled for clarity. Parameters are identical to those of Fig. 1. There are three peaks in  $Q$  at  $H=1$  (single-particle motion), 12 (1/2 subharmonic), and broad peak near  $H=28$  (1/2 subharmonic). Note that  $\bar{\tau}$  is close to  $2T$  when subharmonic modes dominate.

becomes comparable to or even larger than  $T$ , we expect that the motion of the pile can no longer be described by that of a single particle.

To make the idea more quantitative, we measure  $\tau$ . From the time series  $R(t)$ , we locate the times at which  $R(t)$  reaches local minima. The interval between successive minima is defined to be  $\tau$ . One should consider only the flights of initially compact piles. Some of the local minima correspond to a partially expanded state and should not be used. To take that into account, I discard values of  $\tau$  when  $R(t)$  at the launching is larger than 20% of the maximum  $R(t)$ . We calculate distribution  $D(\tau)$  from the resulting set of  $\tau$ . The resulting  $D(\tau)$  seems to be insensitive to a small variation in the cutoff.

When the motion of the pile is periodic, the particles are launched at a specific phase of the vibration. One thus expects that  $D(\tau)$  consists of a few sharp peaks. When the motion becomes more chaotic, the particles are launched at a more irregular phase. The relaxation time  $\tau$  becomes more random and the peaks of  $D(\tau)$  broaden out. Thus one can think of  $D(\tau)$  as a representation of  $P(f)$  in temporal domain. Sharply peaked  $P(f)$  corresponds to sharply peaked  $D(\tau)$ , broad  $P(f)$  to broad  $D(\tau)$ .

We now quantify the dominance of a single mode. We define

$$Q^2 \equiv \frac{1}{\tau_{\max} - \tau_{\min}} \int_{\tau_{\min}}^{\tau_{\max}} (D(\tau) - \bar{D})^2 d\tau, \quad (3)$$

where

$$\bar{D} \equiv \frac{1}{\tau_{\max} - \tau_{\min}} \int_{\tau_{\min}}^{\tau_{\max}} D(\tau) d\tau. \quad (4)$$

Here,  $\tau_{\min}$  ( $\tau_{\max}$ ) is the minimum (maximum) of measured  $\tau$ . The quantity  $Q$  measures deviation from a uniform distribution. If the motion of the pile is periodic,  $D(\tau)$  consists of a few sharp peaks and  $Q$  is large. For a chaotic motion, we expect small  $Q$ . In the top part of Fig. 2, we show  $Q$  measured for the parameters of Fig. 1. The curve consists of

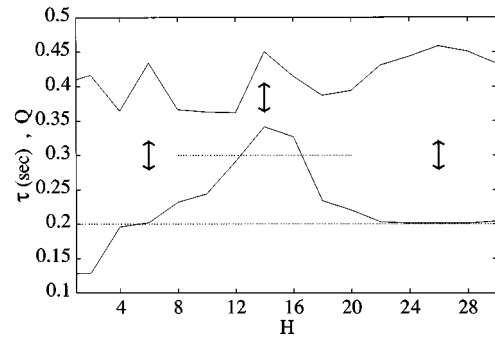


FIG. 3. Quantities  $Q$  (top) and  $\bar{\tau}$  (bottom) are shown for  $\Gamma=3$ . Other parameters are the same as Fig. 2. There are four peaks in  $Q$  at  $H=1$  (single-particle motion), 6 (1/2 subharmonic), 14 (1/3 subharmonic), and around 26 (1/2 subharmonic). Note that  $\bar{\tau}$  is close to  $2T$  ( $3T$ ) when a 1/2 (1/3) subharmonic mode dominates.

three peaks forming a ‘w’ shape. The number of the peaks, and their locations ( $H=1, 12, 28$ ), are what are expected from Fig. 1. Note that the system shows broad resonances around  $H=28$ . Such quality of agreement seems to be typical, which demonstrates the value of  $Q$ . We also calculate the relative contributions of  $P(f_o)$  and  $P(f_o/2)$  to the power spectrum  $P(f)$ , where  $f_o$  is the driving frequency. These contributions also peak around the same locations of the peaks of  $Q$ .

The key quantity that can be calculated from  $D(\tau)$  is  $\bar{\tau}$  [20]. At the bottom of Fig. 2, we show  $\bar{\tau}$  measured using the parameters of Fig. 1. One can see that  $\bar{\tau}$  is indeed larger than the period of the vibration  $T=0.1$  for most  $H$ . Furthermore, notice that the dominances of a single mode seem to occur when  $\bar{\tau}$  is close to an integer multiple of  $T$ , besides small  $H$  (single-particle motion). The dominance near  $H=28$  occurs when  $\bar{\tau}$  is close to  $2T$ , where the frequency of the dominant mode is also  $2T$ . However, the dominance near  $H=12$  occurs near, but not exactly at, where  $\bar{\tau}$  becomes  $2T$  ( $H \sim 10$ ).

Based on these observations, we propose a possible mechanism for the dominance. When the pile, launched from the bottom, comes back to collide with the bottom, there is still significant dispersion, and the pile becomes compact again only after  $\bar{\tau}$ . When  $\bar{\tau}$  is an integer multiple of  $T$ , we expect the pile to repeat the same sequence of motions. Thus  $\bar{\tau}=nT$  is a condition for a single-mode dominance, and the period of the motion is  $nT$ .

The proposed mechanism gets further support by studying the motion for different values of  $\Gamma$ . In Fig. 3, we show  $Q$  and  $\bar{\tau}$  measured for  $\Gamma=3$  while the other parameters remain the same. One can see four dominances: one particle motion at  $H=1$ , two 1/2 subharmonic ( $H=6$ , and around 26) motions, and an one 1/3 subharmonic ( $H=14$ ) motion. Not only the observed periods (1/1, 1/2, and 1/3), but also the order they appear (1/1, 1/2, 1/3, and 1/2, as  $H$  increases) seems to be rather complex. These can be understood easily by looking at the corresponding  $\bar{\tau}$ . The value of  $\bar{\tau}$  is indeed close to  $2T$  near  $H=6, 26$ , where a 1/2 subharmonic mode dominates. However, there is only one dominance of 1/3 subharmonic mode near  $H=14$ , compared to two expected from  $\bar{\tau}$  ( $H \sim 12$  and 16). It is possible that the two nearby

dominances merge to form a single one. As  $\Gamma$  increases to 4 and 5, all of the dominances can again be explained from  $\bar{\tau}$ . No dominance with a period larger than  $3T$  is observed. As  $\Gamma$  is further increased, the dominance becomes much less clear.

We also study the effect of  $f$ . We change  $f$  to 20 and 100 while keeping the other parameters fixed. For  $f=20$  and  $\Gamma=2$ ,  $Q$  assumes a ‘‘w’’ shape as in Fig. 2. The dispersion of the file for  $f=20$  is smaller than that of  $f=10$  at the same  $\Gamma$ . The decrease is more significant for  $f=100$ , where the systematic variation of  $R(t)$  is too small to be seen. This decrease is not unexpected. The expansion of the pile was shown to scale as  $Af$  rather than  $\Gamma$  [9–11]. Therefore, the value of  $Af$  decreases as  $f$  increases for fixed  $\Gamma$ .

We argue that most aspects of the seemingly complex depth dependence can be explained in terms of  $\bar{\tau}$ . However, there are a few remaining questions. First, the origin of the discrepancy on the location of dominances remains unclear. Second, it is not clear why only a  $1/n$  subharmonic mode with integer  $n$  is observed. In principle,  $\bar{\tau}$  can be made suitable for, e.g., a  $2/3$  mode by carefully tuning  $H$ .

Finally, we discuss the conditions for observing the subharmonic motion in an experiment. The amplitude and frequency used in the experiments of Thomas *et al.* are comparable to our simulations [13], but their results are quite different. The difference, we believe, is the coefficient of restitution. The typical value of  $e$  used in the present simulations is less than 0.1, while that of a typical experiment is larger than 0.8 [15]. In order to check the idea, we repeat the simulation with  $e \sim 0.8$  for  $f=10$  and  $\Gamma=2$ . The motion of the pile is indeed chaotic at a small depth, and it becomes more coherent at a larger depth, just as in the experiments. It is possible that the motion of a pile with small  $e$  is similar to that of a taller pile with larger  $e$ . In such a case, subharmonic motion could be observed in a taller pile with large  $e$ .

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- [20] We define  $\bar{\tau} = \tau_o + 2\delta\tau$ , where  $\tau_o$ ,  $\delta\tau$  is the average and standard deviation of the dominant peak of  $D(\tau)$ , respectively. We choose  $\bar{\tau}$  larger than  $\tau_o$  in order to give more weight to the trajectories of initially compact piles. Increasing the difference from the mean ( $2\delta\tau$ ) results in a small but systematic increase of  $\bar{\tau}$ .